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Imbens, G.W.

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POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

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DURATION MODELS WITH TIME-VARYING COEFFICIENTS

Guido W. Imbens

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Duration Models with Time–Varying Coefficients

Guido W. Imbens
Brown University and Tilburg University[1]

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[1]Mailing address: Tilburg University, Postbus 90153, 5000 LE Tilburg, The Netherlands

1 Introduction

The econometric analysis of transition data has been the subject of a large and growing literature since the late seventies. Many of the theoretical concepts in this field, however, are borrowed from the biostatistical literature. This phenomenon is not restricted to the terminology (hazard – and survivor functions are the most obvious examples), but extends to the models employed. One such model is the proportional hazard specification, proposed by Cox [4,5]. It allows one to study the effects of regressors on transition rates without specifying the form of the duration dependence. Econometrics is not biostatistics, however, and the analysis of economic issues brings with it special problems that are not necessarily satisfactorily dealt with by these models. An important econometric innovation was the introduction of unobserved heterogeneity in these models by Lancaster [10]. See also Lancaster [12] and Heckman and Singer [8]. In economics it is often more difficult to control for individual effects than in a controlled hospital environment, where the data for biomedical studies are often obtained.

In this note I want to point out a second implication of the lack of control in economic environments. Not only is it plausible that the population is heterogenous, but it is also likely that the environment in which the population exists, changes over time. In the analysis of panel data this has led researchers to introduce time dummies as well as individual effects. I will propose a convenient way of allowing for time dependent parameters in duration models and suggest with an application to Dutch labor market data the relevance of this approach.

2 The Model

Consider a world in which individuals experience two events, in a particular order. The first might be labelled *entry* or *birth* and the second *exit* or *death*. We are interested in the timing of the second event, given the date that the first event occured, and given other characteristics of the individual which are summarized in a vector x. The duration of the spell between the first and second event will be analyzed primarily in terms of the *hazard function*, or *intensity process*, defined as:

$$(1) \quad \lambda(t, t - t^0, x) = \lim_{\delta \downarrow 0} \Pr\left[t^1 \in [t, t+\delta) \middle| t^1 \geq t, t^0, x\right] / \delta \quad \text{for } t > t^0$$

It is a function of calendar time t, duration $t - t^0$ and characteristics x. One way to analyze models of this type given a dataset[1] $(t_n^1, t_n^0, x_n)_{n=1}^N$ is by specifying a parametric form for the hazard and estimating the parameters by maximum likelihood techniques. The likelihood function would in that case be:

$$(2) \quad \mathcal{L}(\theta) = \prod_{n=1}^{N} \lambda(t_n^1, t_n^1 - t_n^0, x_n; \theta) \exp\left[-\int_{t_n^0}^{t_n^1} \lambda(s, s - t_n^0, x_n; \theta) ds\right]$$

Under standard regularity conditions the maximum likelihood estimator is CAN (consistent and asymptotically normal). Ridder [13], among others, follows this approach. He ignores duration dependence and allows for a flexible calendar time dependence by introducing dummies for two–year intervals. The disadvantage of the model is that the functional form of the hazard function has to be specified completely. Using dummies can to some extent overcome this problem but there is no natural time period in these continous time models, unlike in panel data analysis.

To overcome this heavy reliance on knowledge of the functional form, Cox [4,5] proposed the *proportional hazard model.* In his analysis the hazard depends only on duration $t - t^0$ and characteristics x. Alternatively, one can interpret this as the special case of (1) where all t_n^0 are identical. Cox makes the assumption that the hazard rate can be factorized into a function of duration alone and a function of characteristics alone:

$$(3) \quad \lambda(t, t - t^0, x) = \lambda_0(t - t^0) \cdot \omega(x; \theta)$$

The first factor, λ_0, the ***baseline hazard***, is unknown to the researcher, but the second is known up to a finite dimensional parameter θ. This parameter can be estimated by maximizing the ***partial likelihood function***:

$$(4) \quad \mathcal{L}_p(\theta) = \prod_{n=1}^{N} \omega(x_n; \theta) \Big/ \sum_{m \in R(t_n^1 - t_n^0)} \omega(x_m; \theta)$$

[1] In the discussion in this section the complication of *censored* observations will be ignored for expository reasons. In the Appendix a formal analysis will be given

where the *risk set* R at t consists of those people with a duration not exceeding t:

$$(5) \quad R(t) = \left\{n = 1, 2, \ldots, N \middle| t_n^1 - t_n^0 \leq t\right\}$$

A clear description of the derivation of this partial likelihood function can be found in Cox and Oakes [6] and Lancaster [12]. Consistency and asymptotic normality of this estimator has been proven for various forms of this model. The most popular functional form for ω is log linear:

$$(6) \quad \omega(x;\theta) = \exp(\theta' x)$$

Tsiatis [15] considered this case and gave a proof of the asymptotic properties based on convergence of the average score to a non–stochastic function. The complication in proofs of asymptotic normality lies in the fact that the scores are uncorrelated but not independent. Andersen and Gill [1] extend the proof to the case where the characteristics x are allowed to vary over time. They give a proof in the context of counting processes. The main restriction on the covariate processes x is that they are *predictable* and *locally of bounded variation*. A sufficient condition for this to hold is that the covariate processes are continuous and have bounded first derivatives.

The aim of this paper is to analyze how calendar time can be taken account of in this semi–parametric framework. The first possibility is to add it as a time–varying regressor or *covariate* in the specification of the systematic part of the hazard rate:

$$(7) \quad \lambda(t, t - t^0, x) = \lambda_0(t - t^0) \cdot \omega(t, x; \theta)$$

If we specify

$$(8) \quad \omega(t, x; \theta) = \exp \sum_{k=1}^{K} \theta_k h_k(t, x)$$

with all functions h_k known we are back in the log linear framework analyzed by Andersen and Gill. Examples of such specifications are $h_1(t, x) = x$, $h_2(t, x) = log(t)$ or $h_2(t, x) = t$. Another, more flexible, specification in the spirit of Ridder [13] would be to define some $h_k(t, x) = I[c_k < t \leq c_{k+1}]$.[2]

[2] $I[.]$ is an indicator function, equal to one if the expression between the brackets is true, equal to zero otherwise

If $c_{k+1} - c_k$ were equal to one year, this would amount to introducing yearly dummies in the hazard function.

The above model does not have all the disadvantages of the first, fully parametric model in (2). Nevertheless, it requires the researcher to specify the dependence of the hazard on calendar time completely. The argument why this is difficult in practice is related to the reason behind including calendar time in the first place. The general justification for inclusion is that there might be forces behind the events that are both equal for everybody in the population as well as changing over time. An example is the life span of companies. Irrespective of their characteristics and the date the companies were set up, they might have correlated risks of going bankrupt at the same time via the phase of the business cycle. A similar story can be told for unemployment spells. If the general outlook is bad, the chances of finding a job might be slim for everybody, relative to the chances in good times. If this is the case, one would ideally model these macro processes jointly with the individual behaviour or condition on their paths. However, this requires knowledge and observability of the exact processes that influence the hazard function. If this is not available one should not restrict the time effect to a particular form. Techniques that do not require such a form are to be preferred.

Duration dependence on the other hand is an effect that can potentially be explained within economic models. Jovanovic [9] and Van Den Berg [16] have studied models in which optimizing behaviour by individuals leads to hazard functions with particular forms of duration dependence. In the model studied by Van Den Berg, unemployed individuals will lower their reservation wage over the unemployment spell in anticipation of a decline in benefit levels. This causes the hazard function to be an increasing function of duration. Jovanovic studies, among others, job–to–job transitions. He finds that the hazard should increase initially, when employee and employer learn about the quality (or the lack of quality) of the match, and then decrease, once the match has been found to be a successful one. These examples suggest that it is easier to model duration dependence than calendar time dependence.

This is the reason to propose reversing the roles of calendar time and duration in (7) as the second way to incorporate calendar time dependence. Instead of parametrizing the dependence on calendar time and leaving the dependence on duration free, one could specify the hazard in the following way:

$$(9)\quad \lambda(t, t-t^0, x) = \lambda_0(t)\cdot\omega(t-t^0, x;\theta)$$

with λ_0 an unknown function of time, and ω a known function. In particular one could linearize the second factor again:

$$(10)\quad \omega(t-t^0, x;\theta) = \exp[\theta' h(t-t^0, x)] = \exp\sum_{k=1}^{K}\theta_k h_k(t-t^0, x)$$

Consider the ordered exit times $t^1_{\iota(1)} < t^1_{\iota(2)} < \ldots < t^1_{\iota(N-1)} < t^1_{\iota(N)}$, where $\iota(i)$ gives the identity of the i^{th} individual to exit. Conditional on all the regressors x, the entry dates t^0 and the first exit time $t^1_{\iota(1)}$, the probability of individual n being the first to exit is:

$$(11)\quad \Pr\left[\iota(1) = n \,\middle|\, t^1_{\iota(1)}, x, t^0\right] = 0 \qquad \text{if } t^0_n \geq t^1_{\iota(1)}$$

$$= \frac{\exp[\theta' h(t^1_{\iota(1)} - t^0_n, x_n)]}{\sum_{m\in R(t^1_{\iota(1)})}\exp[\theta' h(t^1_{\iota(1)} - t^0_m, x_m)]} \qquad \text{if } t^0_n < t^1_{\iota(1)}$$

with the risk set $R(t)$ consisting of the people who entered but not exited before t.

$$(12)\quad R(t) = \left\{n = 1, 2, \ldots, N \,\middle|\, t^0_n < t \leq t^1_n\right\}$$

In the same manner we calculate the probability of individual m being the second to exit, given the identity of the first individual to exit, the regressors and entry times. If we proceed in this way till all the exit times are accounted for, we obtain the partial likelihood by multiplying all the probabilities:

$$(13)\quad \mathcal{L}_p(\theta) = \prod_{i=1}^{N}\Pr\left[\iota(i)\,\middle|\, t^1_{\iota(i)}, x, t^0, (\iota(j), j < i)\right]$$

$$= \prod_{i=1}^{N}\frac{\exp[\theta' h(t^1_{\iota(i)} - t^0_{n(i)}, x_{\iota(i)})]}{\sum_{m\in R(t^1_{\iota(i)})}\exp[\theta' h(t^1_{\iota(i)} - t^0_m, x_m)]}$$

$$= \prod_{n=1}^{N} \frac{\exp[\theta' h(t_n^1 - t_n^0, x_n)]}{\sum_{m \in R(t_n^1)} \exp[\theta' h(t_n^1 - t_m^0, x_m)]}$$

where the risk set is again that defined in (12). In the appendix sufficient conditions for the maximand of this function to be a consistent and asymptotically normal estimator of θ are given. The proof is a straightforward application of the Andersen and Gill results.

An estimator for the integrated baseline hazard can be constructed in the same way as proposed by Breslow [2,3] for the original Cox model. Define

$$\Lambda(t, \underline{t}) = \int_{\underline{t}}^{t} \lambda_0(s) ds$$

Then the Breslow estimator for $\Lambda(t, \underline{t})$ is:

$$(14) \quad \hat{\Lambda}(t_j^1, \underline{t}) = \sum_{\underline{t} < t_n^1 \leq t_j^1} \frac{1}{\sum_{m \in R(t_n^1)} \exp[\theta' h(t_n^1 - t_m^0, x_m; \theta)]}$$

for $j = 1, 2, \ldots, N$, and linear interpolation for values of t inbetween two exit times.

3 Testing the Time–Invariance Assumption

Before one engages in the complication of estimating the model with a time–varying baseline hazard function, it might be sensible to test the null hypothesis of a time-invariant baseline hazard. In this section tests are developed for this purpose. They are score tests developed along the same lines as the original tests for neglected heterogeneity. Being score tests they do not require estimation of the model under the alternative hypothesis and they are therefore practical choices.

Under the null hypothesis $\lambda_0(t) = \lambda_0$ for all t. For ease of notation the constant λ_0 will be absorbed in ω. The model can then be estimated by maximizing the logarithm of the likelihood function

$$(15) \quad L(\theta) = \sum_{n=1}^{N} \ln \omega(t_n^1 - t_n^0, x_n; \theta) - \int_{t_n^0}^{t_n^1} \omega(s - t_n^0, x_n; \theta) ds$$

$$= \sum_{n=1}^{N} \ln \omega(t_n^1 - t_n^0, x_n; \theta) - \int_0^{t_n^1 - t_n^0} \omega(s, x_n; \theta) ds$$

In this structure tests are conveniently implemented by testing $\alpha = 0$ in the alternative specification of the hazard function:

(16) $\lambda(t, t - t^0, x) = \omega(t - t^0, x; \theta) \cdot \exp(\alpha h(t))$

for some fully specified function $h(t)$. If ω is log linear, the test is easier to interpret. The hazard function has then the form:

(17) $\lambda(t, t - t^0, x) = \exp\left[\sum_{k=1}^{K} \theta_k h_k(t - t^0, x) + \alpha h(t)\right]$

The test can now be interpreted as one on left out variables. Under the null, only the variables $h_k(t - t_n^0, x)$ should be in the hazard function, while under the alternative the variable $h(t)$ should also be in there.

The score function for the general specification is

(18) $h(t^1) - \int_0^{t^1 - t^0} h(s + t^0) \omega(s, x; \theta) ds$

The first term is the value of the neglected variable at t, the second term gives the integral of the neglected variable over the spell, weighted by the hazard function under the null.

Two examples will be considered. First assume that under the null the model is Weibull. In that case we have:

(19) $\omega(s, x; \theta) = s^{\theta_1} \exp(\theta_0 + \theta_2 \cdot x)$

Also assume that we test the coefficient on $h(t) = t$. The form of the score function is then:

(20) $t^1 - \exp(\theta_0 + \theta_2 \cdot x) \dfrac{(t^1 - t^0)^{\theta_1}}{\theta_1 + 1} \left[t^0 + \dfrac{(t^1 - t^0) \cdot (\theta_1 + 1)}{(\theta_1 + 2)} \right]$

If we further simplify by setting $\theta_1 = 0$, thereby reducing the model to an exponential one, this becomes:

(21) $t^1 - t^0(t^1 - t^0) \exp(\theta_0 + \theta_2 x) - \frac{1}{2}(t^1 - t^0)^2 \exp(\theta_0 + \theta_2 x)$

Because under the null $(t^1 - t^0)\exp(\theta_0 + \theta_2 x)$ has a unit exponential distribution given x and t^0, it can be seen that in this simple case the expectation of the score function is zero.

As the second example consider the case where we specify $h(t) = 1$ for $t \in C \subset \Re$. In that case we test whether the hazard was different from the normal level during a particular period. For the two examples referred to earlier, that of the life span of companies and that of unemployment spells, one could think of testing whether the hazard is different during the boom years of the business cycle. The general form of the score function is in this case:

$$(22) \quad I[t^1 \in C] - \int_C \omega(s - t^0, x; \theta) ds$$

The score function compares the number of people failing in C with the cumulative risk in that period.

Given the score function in (18), the test statistic can take various forms, depending on the estimator for the variance. One, simple form will be given here. Consider the N dimensional vector ι with all elements equal to one, and the matrix $Y(\theta)$, with n^{th} row $y_n(\theta)'$, for $n = 1, 2, \ldots, N$, defined as:

$$(23) \quad y_n(\theta) = \begin{pmatrix} h(t_n^1) - \int_0^{t_n^1 - t_n^0} h(s + t_n^0) \cdot \omega(s; x_n; \theta) ds \\ \frac{1}{\omega(t_n^1 - t_n^0, x_n; \theta)} \frac{\partial \omega}{\partial \theta}(t_n^1 - t_n^0, x_n; \theta) - \int_0^{t_n^1 - t_n^0} \frac{\partial \omega}{\partial \theta}(s, x_n; \theta) ds \end{pmatrix}$$

The second part of the vector $y_n(\theta)$ consists of the contribution of the n^{th} observation to the derivative of the logarithm of the likelihood function (15) at θ. Therefore the elements of $Y(\hat{\theta})'\iota$ are zero apart form the first one. That element is equal to the sum of the score function (18) evaluated at the maximum likelihood estimate $\hat{\theta}$. The test statistic can be written as

$$\iota' Y(\hat{\theta}) \cdot [Y(\hat{\theta})' Y(\hat{\theta})]^{-1} Y(\hat{\theta})' \iota$$

This is equivalent to N times the R^2 of the regression of ι on $y(\hat{\theta})$. Asymptotically it has a $\mathcal{X}_1$ distribution. A useful analogue is the form of the information matrix test proposed by Lancaster [11].

4 An Application

In this section the analysis presented in the first three sections will be applied to data on labor market behavior. A sample of Dutch men was taken[3] and they were questioned about their labor market status between January 1977 and January 1984. Here we only look at employment to unemployment transitions. For those people who were employed in January 1977 we also know when they started that job. We consider two risks or destinations. Someone employed can leave his current job and accept another without an intervening spell of unemployment. Alternatively he can leave his current job and become unemployed. For the purpose of this analysis we consider spells of the first type as censored spells. This does not create any special problems compared with conventional duration models. Spells of the second type are considered complete spells. We consider two regressors. The first is age at the beginning of the spell minus thirty years. One could use age as a time–varying regressor, but the estimation of full maximum likelihood models would be more complicated. For the results this does not matter greatly. The second regressor is an index for education, ranging from -2 to 2. The higher values indicate higher levels of education. The total number of spells used is 455, of which 346 were censored. A complication arises because of the spells that were started before 1977. We condition on the time spend at risk before that date. For easy reference the likelihood on which all the estimations are based, either directly, or via the partial likelihood based on it, is given here:

$$(24)\quad \mathcal{L}(\theta) = \prod_{n=1}^{455} \left[\lambda_0(t_n^1) \cdot \omega(t_n^1 - t_n^0, x_n; \theta)\right]^{d_n}$$

$$\times \exp\left[-\int_{\max(t_n^0, \underline{t})}^{t_n^1} \lambda_0(s) \cdot \omega(s - t_n^0, x_n; \theta) ds\right]$$

where $\underline{t}$ is January 1977, and d_n a censoring indicator, equal to one if t_n^1 is an exit time, and equal to zero otherwise.

[3] The data used here are part of the ORIN (Onderzoek naar Relatievormen in Nederland) dataset. This dataset was set up in 1984 by NIDI (Nederlands Interuniversitair Demografisch Instituut) in cooperation with the Universities of Amsterdam, Tilburg and Wageningen. I wish to thank the NIDI for making these data available to me.

Table 1: The Weibull Model

variable	coefficient	s.d.
constant	-4.90	(0.43)
ln duration	-0.11	(0.10)
age	0.02	(0.01)
education	-0.17	(0.10)

The first model estimated is a Weibull model. The specification is as given in (19). Table one gives the estimation results for this model.

The score test based on the inclusion of an extra regressor t in the logarithm of the hazard, described in detail in the last section was performed. The value obtained was 13.0. The 95% quantile for the appropriate $\mathcal{X}_1$ distribution is 3.8. The test therefore clearly indicates the importance of calendar time. This can also be seen in the following, more informal, but possibly more insightful, way: Consider the model:

$$(25) \quad \lambda(t, t - t^0, x) = \lambda_k \quad \text{for } c_{k-1} < t \leq c_k \quad \text{for } k = 1, 2, \ldots, K$$

where $c_0 = 0$ and $c_K = \infty$. The model is very similar to the piecewise constant hazard as discussed by Cox and Oakes [6] and Lancaster [12]. The difference being that here the constancy is over calendar time –, not over duration intervals. This model was estimated with seven, yearly intervals. In table two the estimates are compared to the unemployment rate in The Netherlands for that year[4]. The rationale for this is that the unemployment rate might be a good indicator for the macro forces that drive the employment to unemployment transition. The yearly hazard rates are not very precisely estimated, as the standard deviations indicate, but there does seem to be a tendency for low values of the hazard to occur at years during which the unemployment rate was also low. This is of course not a surprising result, but it is difficult to explain using the type of models that have conventionally been used for duration analysis. The considerable changes in the unemployment rate in The Netherlands in the years 1977 to

[4]these data are from the handbook of the CBS (Centraal Bureau voor de Statistiek).

Table 2: Yearly Transition and Unemployment Rates

year	hazard	s.d.	U-rate
1977	0.0025	(0.0007)	4.0
1978	0.0018	(0.0068)	3.8
1979	0.0026	(0.0082)	3.6
1980	0.0059	(0.0126)	4.4
1981	0.0069	(0.0140)	7.0
1982	0.0067	(0.0142)	10.0
1983	0.0044	(0.0118)	12.4

Table 3: Proportional Hazard Specification

variable	coefficient	s.d.
ln duration	-0.21	(0.10)
age	0.01	(0.01)
education	-0.19	(0.09)

1983 forcefully drive home the fact that the assumption of a time–invariant environment is unacceptable.

The next model estimated maintains the weibull specification for the ω part of the hazard, but leaves the baseline hazard free. The specification is

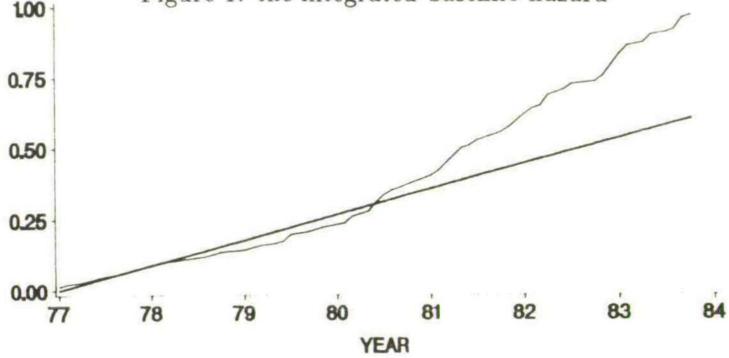
$$(26) \quad \lambda(t, t - t^0, x) = \lambda_0(t) \cdot (t - t^0)^{\theta_1} \exp(\theta_2 x)$$

The estimation results are in table 3. The coefficients on duration are very different for the two models. If we allow the constant term to vary with calendar time, the negative duration dependence becomes much more pronounced.

An additional argument for the approach advocated in this paper can be made. Consider the integrated baseline hazard. If the Weibull specification with a time–invariant intercept is correct, the maximum likelihood estimate is

$$\hat{\Lambda}(t, t) = e^{-4.90} \cdot (t - t)$$

Figure 1: the integrated baseline hazard



Without the assumption that the intercept is constant over time, the integrated hazard can be estimated using Breslow's estimator, given in (14). If this second estimate differs systematically from the first, one has another indication of the inadequacy of the time–invariant model.

Figure 1 gives the Breslow estimator and the estimator under the restriction that λ_0 is a constant. It clearly shows that the slope of the integrated hazard is not constant over time. It increases markedly around 1980. The form of the change also suggests that it would be difficult to model the time effect with a simple continuous function like $h(t) = t$, in (8). This figure makes the interpretation of the bias in the duration dependence easier. Over time the average hazard increases. If one does not take this into account, it appears that the hazard does not decrease very rapidly with duration. The increase with calendar time and the decrease with duration partially cancel each other out if they are not both incorporated in the model.

5 Conclusion

In this paper an alternative form of the proportional hazard model is proposed. It allows one to introduce correlation between exit rates at the same (calendar) time for different individuals. This correlation is explained by assuming a changing macro environment that affects all individuals in a similar way. The assumption that this effect is proportional to the systematic part of the hazard function makes it possible to estimate the parameters of the latter without parametrizing the time effect. One can, in the context of this model, still allow for, and estimate, duration effects. These should be parametrized. These modifications to the original Cox model are possible by reversing the roles of duration and calendar time. It is argued that flexibility with respect to the effects of these macro processes is of parcticular relevance in economic models. An example using Dutch data on labor market behavior illustrates the idea that ignoring calendar time effects can have severe consequences for the estimation of duration dependence. A topic for future research is the effect of unobserved heterogeneity in this model.

A Appendix

In this appendix conditions will be given for consistency and asymptotic normality of the estimator defined as the maximand of (13). The conditions are such that a theorem by Andersen and Gill [1] can be applied. For the exact form of their theorem and the accompanying proof the reader is referred to their paper. One of the conditions is modified in order to make the theorem apply to the commonly used Weibull model.

First the model will be stated in a slightly different version. The hazard rate or intensity process is:

$$(27)\quad \lim_{\delta \downarrow 0} \Pr\left[t^1 \in [t, t+\delta) | t^1 \geq t, t^0, x\right] / \delta = Y(t) \cdot \lambda_0(t) \exp[\theta^{*\prime} h(t - t^0, x)]$$

where $Y(t)$ is an indicator for entry. It is equal to one if $t^0 < t$ and equal to zero otherwise. The functions $h(.,.)$ are known. An example is $h_0(s, x) = s$, $h_1(s, x) = x$.

We are interested in estimating θ^* on the basis of observations of the form $(t_n^0, t_n^1, d_n, x_n)_{n=1}^N$, where d_n is a censoring indicator, equal to one if t_n^1 is an exit time and equal to zero otherwise.

The partial likelihood function is:

(28) $\mathcal{L}_p(\theta) = \prod_{n=1}^{N} \frac{d_n \cdot \exp[\theta' h(t_n^1 - t_n^0, x_n; \theta)]}{\sum_{m \in R(t_n^1)} \exp[\theta' h(t_n^1 - t_m^0, x_m; \theta)]}$

where the risk set $R(t)$ is defined as:

(29) $R(t) = \left\{ n = 1, 2, \ldots, N | t_n^0 < t \leq t_n^1 \right\}$

Define $\hat{\theta}_N$ to be the maximand of (28).

Assumption 1 *(t_n^0, t_n^1, d_n, x_n) and (t_m^0, t_m^1, d_m, x_n) are independent for all $n \neq m$.*

Assumption 2 *$t^0 \in [0, a]$, $x \in X$, with X a compact subset of $\Re^L$, $\theta^* \in \Theta$, with Θ a compact subset of $\Re^K$.*

These assumptions do not require much explanation. They are not the weakest possible, but they fit in with conventional econometric practice.

Assumption 3 *All observations are censored at b if they have not yet exited or been censored before. $0 \leq \lambda_0(t) \leq c$ for all $t \in [0, b]$. $h(s, x)$ is continuous on $(0, b] \times X$.*

The assumption about censoring prevens problems that could occcur with observations containing unbounded information. The assumption about continuity is carefully stated to allow the specification $h(s, x) = \ln(s)$, which leads to the Weibull model.

To ensure identification and asymptotic normality one has to look at the first and second moments of h. Define:

$$S^{(0)}(\theta, t) = \frac{1}{N} \sum_{n=1}^{N} Y_n(t) \cdot \exp[\theta' h(t - t_n^0, x_n)]$$

$$S^{(1)}(\theta, t) = \frac{1}{N} \sum_{n=1}^{N} h(t - t_n^0, x_n) \cdot Y_n(t) \cdot \exp[\theta' h(t - t_n^0, x_n)]$$

$$S^{(2)}(\theta, t) = \frac{1}{N} \sum_{n=1}^{N} h(t - t_n^0, x_n \cdot h(t - t_n^0, x_n)' \cdot Y_n(t) \exp[\theta' h(t - t_n^0, x_n)]$$

$$\text{and} \quad \Sigma(\theta) = \int_0^t \left[\frac{S^{(2)}(\theta,t)}{S^{(0)}(\theta,t)} - \frac{S^{(1)}(\theta,t)}{S^{(0)}(\theta,t)} \cdot \frac{S^{(1)}(\theta,t)'}{S^{(0)}(\theta,t)} \right] \cdot S^{(0)}(\theta,t)\lambda_0(t)dt$$

Assumption 4 *$S^{(0)}$, $S^{(1)}$ and $S^{(2)}$ converge to their expectation uniformly in θ and t. $\Sigma(\theta^*)$ is positive definite.*

Assumption 5 *For all $\epsilon > 0$:*

$$\sup_x \int_0^t |h(s,x)| \cdot I[N^{-1/2} \cdot |h(s,x)| > \epsilon] \exp[\theta' h(s,x)] ds \xrightarrow{P} 0$$

The last assumption is a slighly modified version of condition C in Andersen and Gill [1]. In both cases it is trivially fulfilled if h is bounded. The advantage of the formulation here is that the Weibull specification is included. To see this, let $h(s,x) = \ln(s)$, and assume that the coefficient is $\theta > -1$. Then:

$$\int_0^t |\ln(s)| \cdot I[N^{-1/2}|\ln(s)| > \epsilon] \exp[\theta \ln(s)] ds$$

$$= -\int_0^{a(N)} \ln(s) \cdot s^\theta ds$$

for $a(N) = \exp[-N^{1/2} \cdot \epsilon]$. This is equal to

$$-\int_0^{a(N)} \ln(s) \cdot s^{\frac{1}{2}+\frac{\theta}{2}} \cdot s^{\frac{\theta}{2}-\frac{1}{2}} ds$$

which is, for N large enough, and therefore $a(N)$ small enough, bounded by

$$\int_0^{a(N)} s^{\frac{\theta}{2}-\frac{1}{2}} ds$$

This goes to zero as $a(N)$ goes to zero, which shows that assumption (5) is satisfied.

Theorem 1 *Suppose assumptions 1–5 are satisfied. Then, as $N \longrightarrow \infty$,*

1. $\hat{\theta} \xrightarrow{P} \theta^*$
2. $\sqrt{N}(\hat{\theta} - \theta^*) \xrightarrow{d} \mathcal{N}(0, \Sigma(\theta^*)^{-1})$

proof: we check the conditions for lemma 3.1 and theorem 3.2 in Andersen and Gill with the interval $[0,1]$ replaced by $[0,\underline{t}]$. Conditions A, B and D follow trivially from our assumptions. Condition C is only neccessary to guarantee that as $N \longrightarrow \infty$,

$$\int_0^{\underline{t}} \frac{1}{N} \sum_{n=1}^{N} |h| \cdot I[N^{-1/2} \cdot |h| > \epsilon] Y_n(t) \lambda_0(t) \exp[\theta' h] dt \xrightarrow{P} 0$$

for all ϵ. This follows from assumption 5. $\mathcal{QED}$.

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